Complete semi-conjugacies for psuedo-Anosov homeomorphisms

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Abstract

Suppose S is a surface of genus ≥ 2, $f : S \to S$ is a surface homeomorphism isotopic to a pseudo-Anosov map α and suppose $\tilde{S}$ is the universal cover of S and F and A are lifts of f and α respectively. A result of A. Fathi shows there is a semiconjugacy $\Theta : \tilde{S} \to \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u$ from F to $\bar{A}$, where $\bar{\mathcal{L}}^s$ ($\bar{\mathcal{L}}^u$) is the completion of the R-tree of leaves of the stable (resp. unstable) foliation for A and $\bar{A}$ is the map induced by A.

We generalize a result of Markovich and show that for any $g \in \text{Homeo}(S)$ that commutes with f and is isotopic to the identity with identity lift G and for any (c, w) in the image of Θ each component of $\Theta^{-1}(c, w)$ is G-invariant.

1 Introduction

Suppose that S is a closed surface and that $\alpha : S \to S$ is either an orientation preserving linear Anosov map of T^2 or an orientation preserving pseudo-Anosov homeomorphism of a higher genus surface. In the former case note that α fixes the point e that is the image of $(0, 0)$ in the usual projection of $\mathbb{R}^2$ to T^2. The first author [Fr] proved that if $f \in \text{Homeo}(T^2)$ is isotopic to α and fixes e then there is a unique map $p : T^2 \to T^2$ that fixes e, is isotopic to the identity and that semi-conjugates f to α; i.e $pf = \alpha p$.

To describe this case further we work in the universal cover $\mathbb{R}^2$ of T^2. Let A, F and P be the lifts of α, f and p respectively that fix $(0, 0)$ and note that $PF^k = A^k P$ for all $k \in \mathbb{Z}$ (because $PF^k((0, 0)) = A^k F((0, 0))$. We say that the F-orbit of $\tilde{y} \in \mathbb{R}^2$ *shadows* the A-orbit of $\tilde{x} \in \mathbb{R}^2$ if $\tilde{d}(F^k(\tilde{y}), A^k(\tilde{x})) \leq C$ for some C and all $k \in \mathbb{Z}$; we also say that the f-orbit of $y \in T^2$ *globally shadows* the α-orbit of $x \in T^2$. Since p is homotopic to the identity, there exists $C > 0$ so that $\text{dist}(P(\tilde{y}), \tilde{y}) < C$ for all

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$\tilde{y} \in \mathbb{R}^2$. In particular, $\operatorname{dist}(F^k(\tilde{y}), A^k(P(\tilde{y})) = \operatorname{dist}(F^k(\tilde{y}), PF^k(\tilde{y})) < C$ for all k so the F-orbit of $\tilde{y}$ shadows the A-orbit of $P(\tilde{y})$ for all $\tilde{y} \in \mathbb{R}^2$. It is well known that no two A-orbits shadow each other so P is completely determined by this shadowing property. The surjectivity of P reflects the fact that every A-orbit is shadowed by some F-orbit. The fact that P is defined on all of $\mathbb{R}^2$ reflects the fact that every F-orbit shadows some A-orbit.

Suppose now that α is pseudo-Anosov, that f is isotopic to α and that $A : \tilde{S} \to \tilde{S}$ is a lift of α. The isotopy between α and f lifts to an isotopy between A and a lift $F : \tilde{S} \to \tilde{S}$ of f. Equivalently, F is the unique lift of f that induces the same action on covering translations as A. Let $d(x, y)$ be any path metric on S and let $\tilde{d}(\tilde{x}, \tilde{y})$ be its lift to $\tilde{S}$. Shadowing in $\tilde{S}$ and global shadowing in S are defined as above using $\tilde{d}$ in place of the Euclidean metric on $\mathbb{R}^2$. It is not hard to construct examples (c.f. Proposition 2.1 of [H2]) for which there are F-orbits that are not shadowed by any A-orbit. On the other hand, the second author proved [H1] that every A-orbit is shadowed by some F-orbit. More precisely, there exists a closed f-invariant set $Y \subset S$ with full pre-image denoted $\tilde{Y} \subset \tilde{S}$ and a continuous equivariant surjection $P : \tilde{Y} \to \tilde{S}$ such that $PF = AP$.

The F-orbits of $\tilde{S} \setminus \tilde{Y}$ do not shadow A-orbits. It is natural to ask if there is some larger context in which one can understand these orbits. Fathi [Fa] answered this by considering leaf spaces of the stable and unstable foliations.

In the Anosov case, the hyperbolic linear map A has stable and unstable invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ consisting of straight lines parallel to the eigenvectors of A. Let $\mathcal{L}^s$ and $\mathcal{L}^u$ be the leaf spaces of $\mathcal{F}^s$ and $\mathcal{F}^u$ respectively. Then $\mathcal{L}^s$ and $\mathcal{L}^u$ can be identified with $\mathbb{R}$ in such a way that A induces homotheties $A^s : \mathcal{L}^s \to \mathcal{L}^s$ and $A^u : \mathcal{L}^u \to \mathcal{L}^u$ defined respectively by $x \to \lambda x$ and by $x \to x/\lambda$ where $\lambda > 1$ is an eigenvalue of A. Thus $(A^s, A^u) : \mathcal{L}^s \times \mathcal{L}^u \to \mathcal{L}^s \times \mathcal{L}^u$ is naturally identified with $A : \mathbb{R}^2 \to \mathbb{R}^2$ and the shadowing argument that defines P can be done in this product of leaf spaces.

In the pseudo-Anosov case the stable and unstable foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ have singularities and their leaf spaces $\mathcal{L}^s$ and $\mathcal{L}^u$ are more complicated. Namely, $\mathcal{L}^s$ and $\mathcal{L}^u$, and their metric completions $\bar{\mathcal{L}}^s$ and $\bar{\mathcal{L}}^u$ have the structure of $\mathbb{R}$-trees [MS]. As in the previous case A induces a homothety $A^s : \bar{\mathcal{L}}^s \to \bar{\mathcal{L}}^s$ that uniformly expands distance by a factor $\lambda > 1$ and a homothety $A^u : \bar{\mathcal{L}}^u \to \bar{\mathcal{L}}^u$ that uniformly contracts distance by the factor $1/\lambda$. Let $\bar{A} = (A^s, A^u) : \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u \to \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u$.

Let $Q_s : \tilde{S} \to \mathcal{L}^s$ be the natural map that sends a point in $\tilde{S}$ to the leaf of $\mathcal{F}^s$ that contains it. Define Q_u similarly and let $Q = Q_s \times Q_u : \tilde{S} \to \mathcal{L}^s \times \mathcal{L}^u$. Denote the subset of $\mathcal{L}^s \times \mathcal{L}^u$ consisting of pairs of leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ that have a point in common by Δ. Then $Q(\tilde{S}) = \Delta$, and $Q : \tilde{S} \to \Delta$ is a homeomorphism. Moreover,

the following diagram commutes.

$$\begin{tikzcd}
\tilde{Y} \arrow[r, "F"] \arrow[d, "P"'] & \tilde{Y} \arrow[d, "P"] \\
\tilde{S} \arrow[r, "A"] \arrow[d, "Q"'] & \tilde{S} \arrow[d, "Q"] \\
\Delta \arrow[r, "\bar{A}|_\Delta"'] & \Delta
\end{tikzcd}$$

Fathi [Fa] extended QP to a non-surjective map $\Theta : \tilde{S} \to \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u$ that makes the following diagram commutes.

$$\begin{tikzcd}
\tilde{S} \arrow[r, "F"] \arrow[d, "\Theta"'] & \tilde{S} \arrow[d, "\Theta"] \\
\bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u \arrow[r, "\bar{A}"'] & \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u
\end{tikzcd}$$

See also [RHU] for more details on the map Θ.

The maps $p : T^2 \to T^2$ and $\Theta : S \to \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u$ depend canonically on f and so determine canonical decompositions $\{p^{-1}(x)\}$ of T^2 and $\{\Theta^{-1}(c, w)\}$ of S. If g commutes with f then one expects g to preserve this decomposition. If g is isotopic to the identity then one might even expect g to setwise preserve each element of the decomposition.

Implicit in [Mark] is an even stronger and more surprising fact for the Anosov case. Namely that if g commutes with f and is isotopic to the identity then g setwise preserves each *component* of the decomposition. It is not hard to see that this decomposition is upper semi-continuous and that each element is cellular.

Theorem 1.1. (Markovich) *Suppose that $\alpha : T^2 \to T^2$ is a linear Anosov map, that $f \in \mathrm{Homeo}(T^2)$ is isotopic to α and fixes e and that $p : T^2 \to T^2$ is the unique map that fixes e and satisfies $pf = \alpha p$. If $g \in \mathrm{Homeo}(T^2)$ commutes with f, is isotopic to the identity and setwise preserves $p^{-1}(e)$ then each component of $p^{-1}(x)$ is g-invariant for all $x \in T^2$.*

The main result of this paper is the following extension of Markovich's result to the pseudo-Anosov case. If g is isotopic to the identity then the *identity lift* G of g is the unique lift that commutes with all covering translations.

Theorem 1.2. *Suppose that $\alpha : S \to S$ is pseudo-Anosov, that $f \in \mathrm{Homeo}(S)$ is isotopic to α and that $A, F : \tilde{S} \to \tilde{S}$ and $\Theta : \tilde{S} \to \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u$ are as above. If $g \in \mathrm{Homeo}(S)$ commutes with f and is isotopic to the identity then the identity lift $G : \tilde{S} \to \tilde{S}$ of g commutes with F and preserves each component of $\Theta^{-1}(c, w)$ for all $(c, w) \in \Theta(\tilde{S})$.*

Our proof of Theorem 1.2 makes use of arguments from [Mark]. We give complete details for the reader's convenience and because the arguments from [Mark] are not easily referenced. It is straightforward to modify our proof of Theorem 1.2 to obtain a proof of Theorem 1.1. We leave that to the interested reader.

2 Proof of Theorem 1.2

We assume throughout this section that $\alpha : S \to S$ is pseudo-Anosov with expansion factor $\lambda > 1$, that $f \in \mathrm{Homeo}(S)$ is isotopic to α, and that $A : \tilde{S} \to \tilde{S}$ and $F : \tilde{S} \to \tilde{S}$ are lifts of α and f that induce the same action on covering translations.

The transverse measures on the lifts $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ of the stable and unstable measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ for α determine pseudo-metrics d^u and d^s on $\tilde{S}$ such that

$$d^u(A(x), A(y)) = \lambda d^u(x, y) \text{ and } d^s(A(x), A(y)) = \lambda^{-1} d^s(x, y). \tag{1}$$

for all $x, y \in \tilde{S}$. Moreover, $d^u(x, y) = 0$ [resp. $d^s(x, y) = 0$] if and only if x and y belong to the same leaf of $\tilde{\mathcal{F}}^s$ [resp. $\tilde{\mathcal{F}}^u$]. There is also [B] a singular Euclidean metric d on $\tilde{S}$ such that

$$d(x, y) = \sqrt{d^s(x, y)^2 + d^u(x, y)^2}$$

on each standard Euclidean chart (i.e. one without singularities). In particular, $d(x, y) \leq 2 \max\{d^s(x, y), d^u(x, y)\}$.

For any $x, y \in \tilde{S}$ there is a unique (up to parametrization) path ρ with endpoints x and y and with length equal to $d(x, y)$. Subdividing ρ at the singularities that it intersects decomposes ρ into a concatenation of linear subpaths. We will refer to ρ as the *geodesic joining x to y*. If the intersection of two geodesics γ_1 and γ_2 is more than a single point, then it is a path whose endpoints are either singularities or endpoints of γ_1 or γ_2. Any leaf without singularities of either foliation is a geodesic, as is any embedded copy of $\mathbb{R}$ in a leaf with singularities.

Remark 2.1. Since the geodesic γ joining x to y (minimizing length as measured by d) consists of a finite collection of segments whose interiors lie in Euclidean charts we can conclude $\max\{d^s(x, y), d^u(x, y)\} \leq d(x, y) \leq 2 \max\{d^s(x, y), d^u(x, y)\}$.

We denote the leaf of $\tilde{\mathcal{F}}^s$ that contains x by $W^s(x)$, the leaf space of $\tilde{\mathcal{F}}^s$ by $\mathcal{L}^s$ and the image of $W^s(x)$ in $\mathcal{L}^s$ by $Q_s(x)$). Thus $Q_s^{-1}(Q_s(x)) = W^s(x)$. Moreover, d^u induces a metric on $\mathcal{L}^s$ (which we also denote d^u) by setting $d^u(Q_s(x_1), Q_s(x_2)) = d^u(x_1, x_2)$. This is easily seen to be independent of the choice of x_1 and x_2. This

metric gives $\mathcal{L}^s$ the structure of an $\mathbb{R}$-tree [MS]. As illustrated in Remark 2.2, $\mathcal{L}^s$ is not complete. We denote the metric completion by $(\bar{\mathcal{L}}^s, d^u)$

Remark 2.2. One can construct a non-converging Cauchy sequence $L_i = Q_s(x_i)$ in $\mathcal{L}^s$ as follows. Choose a singularity $\tilde{x}_0 \in \tilde{S}$, a stable ray R_0 initiating at x_0 and a sequence $\epsilon_i > 0$ whose sum is finite. Assuming inductively that x_i and R_i have been defined, choose a singularity $x_{i+i} \in \tilde{S}$ such that $W^u(x_{i+1}) \cap R_i \neq \emptyset$ and such that $d^u(x_{i+1}, x_i) < \epsilon_{i+1}$. Let R_{i+1} be a stable ray in $W^s(x_{1+1})$ that initiates at x_{i+1} and whose interior is contained in a component of the complement of $W^u(x_{i+1})$ that is disjoint from x_i. By construction, $d^u(L_i, L_j) = \epsilon_{i+1} + \cdots + \epsilon_j$ for all $i \leq j$ and there are no accumulation points of the $W^s(x_i)$'s. The former shows that the L_i's are a Cauchy sequence and the latter implies that this sequence is non-convergent.

Similarly $\mathcal{L}^u$, the space of unstable leaves in $\tilde{S}$ is an $\mathbb{R}$-tree with metric d^s and $Q_u : \tilde{S} \to \mathcal{L}^u$ satisfies $Q_u^{-1}(Q_u(x)) = W^u(x)$. The metric completion of $\mathcal{L}^u$ is $(\bar{\mathcal{L}}^u, d^s)$ and we use the metric $\bar{d} = \max\{d^u, d^s\}$ on $\bar{\mathcal{L}}^u \times \bar{\mathcal{L}}^s$. Define $Q : \tilde{S} \to \mathcal{L}^u \times \mathcal{L}^s$ by $Q(x) = (Q^s(x), Q^u(x))$ and let $\Delta = Q(\tilde{S})$ equipped with the subset topology. Note that Δ can be characterized as the subset of $\mathcal{L}^s \times \mathcal{L}^u$ consisting of pairs of leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ that have a point in common.

The pseudo-Anosov map A on $\tilde{S}$ induces a homeomorphism $A_s : \mathcal{L}^s \to \mathcal{L}^s$. From the properties of d^s (equation (1) above) it is clear that A_s is an expanding homothety, i.e. for any $L_1, L_2 \in \mathcal{L}^s$

$$d^u(A_s(L_1), A_s(L_2)) = \lambda d^u(L_1, L_2). \tag{2}$$

The map $A_u : \mathcal{L}^u \to \mathcal{L}^u$ is defined analogously and has similar properties. Likewise if T is a covering translation on $\tilde{S}$ it induces an isometry $T_s : \mathcal{L}^s \to \mathcal{L}^s$. It is clear that A_s and A_u extend to expanding and contracting homotheties respectively of $\bar{\mathcal{L}}^s$ and $\bar{\mathcal{L}}^u$ which we also denote by A_s and A_u. Also T_s extends to an isometry of $\bar{\mathcal{L}}^s$. The map $T_u : \mathcal{L}^u \to \mathcal{L}^u$ is defined analogously and has similar properties. Let $\bar{A} : \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u \to \bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u$ be the map $A_s \times A_u$.

As a consequence of Fathi's theorem we have the following.

Proposition 2.3. *There exists a unique continuous map $\theta_s : \tilde{S} \to \bar{\mathcal{L}}^s$ satisfying*

(i) There is a commutative diagram

$$\begin{tikzcd} \tilde{S} \arrow[r, "F"] \arrow[d, "\theta_s"'] & \tilde{S} \arrow[d, "\theta_s"] \\ \bar{\mathcal{L}}^s \arrow[r, "A_s"'] & \bar{\mathcal{L}}^s \end{tikzcd}$$

(ii) There exists $C_1 > 0$ such that $d^u(Q_s(x), \theta_s(x)) < C_1$ for all $x \in \tilde{S}$.

(iii) For all $x \in \tilde{S}$, $\theta_s(x)$ is the unique point in $\bar{\mathcal{L}}^s$ with the property that there exists $C_2 > 0$ such that $d^u(Q_s(F^k(x)), A_s^k\theta_s(x)) < C_2$ for all $k \geq 0$.

(iv) $\theta_s(T(x)) = T_s(\theta_s(x))$ for all $x \in \tilde{S}$ and each covering transformation T.

(v) The image of θ_s contains $\mathcal{L}^s$.

Proof. Let $\theta_s : \tilde{S} \to \bar{\mathcal{L}}^s$ be the composition of Θ with the projection of $\bar{\mathcal{L}}^s \times \bar{\mathcal{L}}^u$ onto its first factor. Then properties (i)-(v) follow from the corresponding properties of Θ as proved in [Fa]. □

Let $\theta_u : \tilde{S} \to \bar{\mathcal{L}}^u$ be the analogous semiconjugacy from F to $A_u : \bar{\mathcal{L}}^u \to \bar{\mathcal{L}}^u$. Note then that

$$\Theta(x) = (\theta_u(x), \theta_s(x)).$$

Remark 2.4. Proposition 2.3 can be proved directly as follows. Given $x \in \tilde{S}$ let $z_k = A^{-k}F^k(x)$ and let $L_k = Q_s(z_k)$. Then it is straight forward to show that $\{L_k\}$ is a Cauchy sequence in the d^u metric on $\bar{\mathcal{L}}^s$. It's limit can be taken as the definition of $\theta_s(x)$. Property (i) is then immediate and it is not difficult to show θ_s is continuous and satisfies the other properties.

We denote open ϵ-neighborhoods by $N_\epsilon(\cdot)$. By item (iii) of Proposition 2.3 applied with $k = 0$ there is a constant C_2 such that

$$d^u(Q_s(x), \theta_s(x)) < C_2 \tag{3}$$

for all $x \in \tilde{S}$. Since $d^u(A(x), F(x))$ is bounded independently of x and since Q_s preserves d^u we may also assume that

$$d^u(Q_sF(x), A_sQ_s(x)) = d^u(Q_sF(x), Q_sA(x)) < C_2. \tag{4}$$

Choose $C > \max(C_2, C_2/(\lambda - 1))$ and note that

$$\lambda C - C_2 > C \tag{5}$$

Lemma 2.5. *$\Theta^{-1}(N_\epsilon(c) \times N_\epsilon(w))$ is a bounded subset of $\tilde{S}$ for all $\epsilon > 0$ and all $(c, w) \in \mathcal{L}^s \times \mathcal{L}^u$.*

Proof. For $x \in Q_s^{-1}(c)$ and $z \in \Theta^{-1}(N_\epsilon(c) \times N_\epsilon(w))$ we have

$$\begin{aligned} d^u(x, z) &= d^u(Q_s(x), Q_s(z)) \\ &\leq d^u(Q_s(x), \theta_s(z)) + d^u(\theta_s(z), Q_s(z)) \\ &\leq \epsilon + C_2. \end{aligned}$$

Symmetrically, if $\theta_u(y) = w$ then

$$d^s(y, z) \leq \epsilon + C_2.$$

It follows that $d(z, z') \leq 2\max\{d^u(z, z'), d^s(z, z')\} \leq 2(\epsilon + C_2)$ for all $z, z' \in \Theta^{-1}(N_\epsilon(c) \times N_\epsilon(w))$. □

Lemma 2.6. *If $V(c,\epsilon)$ is defined to be $\theta_s^{-1}(N_\epsilon(c))$, then*

(i) The set $V(c,\epsilon)$ is an open, connected, simply connected, unbounded set for all $c \in \mathcal{L}^s$ and all $\epsilon > 0$.

(ii) If $G : \tilde{S} \to \tilde{S}$ commutes with F and if there is a constant C_1 such that $d(x, G(x)) < C_1$ for all $x \in \tilde{S}$ then there is a ray R that is properly embedded in $\tilde{S}$ such that $R, G(R) \subset V(c,\epsilon)$ and such that R is properly homotopic to $G(R)$ in $V(c,\epsilon)$; i.e. there is a one parameter family R_t of rays in $V(c,\epsilon)$ such that $R_0 = R$ and $R_1 = G(R)$ and such that each R_t is properly embedded in $\tilde{S}$.

Proof. Define
$$Y_k = \{x \in \tilde{S} : d^u(x, W^s(A^k(c)) < \lambda^k\epsilon - C\}$$
or equivalently
$$Y_k = Q_s^{-1}(N_{\lambda^k\epsilon - C}(A_s^k(c))).$$
Then Y_k is an open convex subset of $\tilde{S}$ that is a union of leaves of $\mathcal{F}^s$. In particular, Y_k is an open, connected, simply connected, unbounded set. If $R_1(t)$ and $R_2(t)$ are any two rays in Y_k that are properly embedded in $\tilde{S}$ and if there is a constant C_0 such that $d(R_1(t), R_2(t)) \le C_0$ for all t, then these rays are properly homotopic in Y_k (by a homotopy along geodesics).

Define
$$X_k = F^{-k}(Y_k).$$
Thus
$$\begin{aligned} X_k &= F^{-k}Q_s^{-1}(N_{\lambda^k\epsilon - C}(A_s^k(c))) \\ &= \{x \in \tilde{S} : d^u(Q_sF^k(x), A_s^k(c)) < \lambda^k\epsilon - C\}. \end{aligned}$$
Since F is a homeomorphism, each X_k is an open, connected, simply connected, unbounded set. Also if $R_1(t)$ and $R_2(t)$ are properly embedded rays in $\tilde{S}$, contained in X_k, for which there is a constant C_0 such that $d(R_1(t), R_2(t)) \le C_0$, then these rays are properly homotopic in X_k.

If $x \in X_k$ then Equation (3) implies that $A_s^k\theta_s(x) = \theta_sF^k(x) \in N_{\lambda^k\epsilon}(A_s^k(c))$ and so $\theta_s(x) \in N_\epsilon(c)$ by Equation (2). This proves that $X_k \subset V(c,\epsilon)$. Moreover, by the triangle inequality and Equations (4), (2) and (5) we have
$$\begin{aligned} d^u(Q_sF^{k+1}(x), A_s^{k+1}(c)) &= d^u(Q_sFF^k(x), A_sA_s^k(c)) \\ &\le d^u(A_sQ_sF^k(x), A_sA_s^k(c)) + C_2 \\ &= \lambda d^u(Q_sF^k(x), A_s^k(c)) + C_2 \\ &\le \lambda(\lambda^k\epsilon - C) + C_2 \\ &= \lambda^{k+1}\epsilon - (\lambda C - C_2) \\ &\le \lambda^{k+1}\epsilon - C \end{aligned}$$

which proves that $X_k \subset X_{k+1}$.

If $\theta_s(w) \in N_\epsilon(c)$ then we may choose $\delta < \epsilon$ and $k \geq 1$ so that $\theta_s(w) \in N_\delta(c)$ and so that $\lambda^k \delta < \lambda^k \epsilon - C_2 - C$. Then

$$\theta_s F^k(w) = A_s^k \theta_s(w) \in N_{\lambda^k \epsilon - C - C_2}(A_s^k(c))$$

and Equation (3) implies that

$$Q_s F^k(w) \in N_{\lambda^k \epsilon - C}(A_s^k(c))$$

and hence that $w \in X_k$. We have now shown that $V(c, \epsilon)$ is the increasing union of open, connected, simply connected and unbounded sets X_k thereby completing the proof of (i).

Choose $k \geq 1$ so that $C_1 < \lambda^k \epsilon - C$ and let R' be a ray in $W^s(A_s^k(c))$. Then R' and $G(R'))$ are properly embedded in $\tilde{S}$ and are contained in Y_k and $R := F^{-k}(R')$ and $G(R) = F^{-k} G(R')$ are properly embedded in $\tilde{S}$ and are contained in X_k. Choosing C_0 so that $d(G(x), x) \leq C_0$ for all x, we conclude $d(R(t), G(R(t))) \leq C_0$ and as noted above, R and $G(R)$ are properly homotopic in $X_k \subset V(c, \epsilon)$. This proves (ii). $\square$

The following proof is an adaptation of one that appears in [Mark].

Proof of Theorem 1.2 Let $G : \tilde{S} \to \tilde{S}$ be the unique lift of g that is equivariantly isotopic to the identity; equivalently G is the lift that commutes with all covering translations T of $\tilde{S}$. The commutator $[F, G]$, which must be a covering translation since it is a lift of the identity on S, commutes with all covering translations and hence is the identity. It follows that F and G commute.

Since S is compact and $\bar{d}(Q(G(Ty)), Q(Ty)) = \bar{d}(Q(G(y)), Q(y))$ for all $y \in \tilde{S}$ and all covering translations T, there is a constant C' such that $\bar{d}(Q(G(y)), Q(y)) < C'$ for all $y \in \tilde{S}$. It follows that

$$\begin{aligned}
\bar{d}(QF^k(x), \bar{A}^k(\Theta(G(x))) &\leq \bar{d}(QF^k(x), QF^k(G(x)) + \bar{d}(QF^k(G(x)), \bar{A}^k\Theta(G(x))) \\
&= \bar{d}(QF^k(x), QG(F^k(x)) + \bar{d}(QF^k(G(x)), \bar{A}^k\Theta(G(x))) \\
&\leq C' + \bar{d}(QF^k(G(x)), \bar{A}^k\Theta(G(x))) \\
&\leq C' + \overline{C_2}
\end{aligned}$$

where $\overline{C_2}$ is the maximum of the constants produced by item (iii) of Proposition 2.3 applied to θ_s and to θ_u. The uniqueness part of Proposition 2.3-(iii) therefore implies that $\Theta G = \Theta$. In particular, $\Theta^{-1}(c, w)$ is G-invariant. It suffices to show that each component of $\Theta^{-1}(c, w)$ is G-invariant.

Choose $\epsilon_n \to 0$. Let $V_n = V(c, \epsilon_n)$ be as in Lemma 2.6 and let $H_n = H(w, \epsilon_n)$ be the open set obtained by applying Lemma 2.6 with $\mathcal{L}^s$ replaced by $\mathcal{L}^u$ and θ_s replaced by θ_u. Then $V_n \cap H_n = \Theta^{-1}(N_\epsilon(c) \times N_\epsilon(w))$ is bounded by Lemma 2.5 and

$$\Theta^{-1}(c, w) = \cap_{n=1}^\infty (V_n \cap H_n).$$

Moreover if Λ is a component of $\Theta^{-1}(c,w)$ and we let K_n be the component of $V_n \cap H_n$ containing Λ then $\overline{K_{n+1}} \subset K_n$ for all n where $\overline{K_{n+1}}$ denotes the closure of K_{n+1}. In particular,

$$\cap_{n=1}^{\infty} K_n = \cap_{n=1}^{\infty} \overline{K_{n+1}}.$$

Since each $\overline{K_{n+1}}$ is compact, $\cap_{n=1}^{\infty} K_n$ is non-empty and connected and hence equal to Λ. It therefore suffices to show that each K_n is G-invariant. In fact it is enough to prove that $G(\bar{K}_n) \cap \bar{K}_n \neq \emptyset$, because then $G(\bar{K}_n)$ and $\bar{K}_n$ are both subsets of K_{n-1} and hence $G(K_{n-1}) = K_{n-1}$.

If there were infinitely many components of $V_n \cap H_n$ that contain an element of $\Theta^{-1}(c,w)$ then there would be a sequence $\{x_i\} \subset \Theta^{-1}(c,w)$ converging to some $x \in \Theta^{-1}(c,w)$ and with each x_i in a different component of $V_n \cap H_n$. Since these components are disjoint, x is also an accumulation point of the frontiers of these components. Hence, since the frontier of a component of $V_n \cap H_n$ is contained in the union of the frontier of V_n and the frontier of H_n there is a sequence $\{y_i\}$ converging to x with each y_i in the frontier of either V_n or H_n. This contradicts the continuity of Θ and the fact that $\bar{d}(\Theta(y_i), \Theta(x_i)) = \epsilon_n > 0$ for all i. We conclude that there are only finitely many components of $V_n \cap H_n$ that contains an element of $\Theta^{-1}(c,w)$. Since G commutes with Θ, $G^i(K_n)$ is a such a component for all $i \geq 0$. Thus $G^m(K_n) = K_n$ for some smallest $m \geq 1$.

Let S^2 denote the one point compactification of $\tilde{S}$ obtained by adding a point ∞. The set $V_n \subset \tilde{S}$ can be thought of as a subset of S^2 and when we do so we refer to it simply as V. By Lemma 2.6, V is open, connected and simply connected and so has a prime end compactification $\widehat{V}$. For our purposes the key properties are:

(i) $\widehat{V}$ is topologically a disk D whose interior is identified with V.

(ii) The function $G|_V$ extends continuously to a homeomorphism $\widehat{G} : D \to D$.

(iii) For each continuous arc $\gamma : [0,1] \to S^2$ with $\gamma([0,1)) \subset V$ and $\gamma(1)$ in the frontier of V there is a continuous arc $\widehat{\gamma} : [0,1] \to D$ with $\widehat{\gamma}(t) = \gamma(t)$ for $t \in [0,1)$. The point $\gamma(1)$ is called an *accessible point* of the frontier of V and $\widehat{\gamma}(1)$ is a prime end corresponding to it (there may be more than one prime end corresponding to an accessible point).

(iv) If γ_t is a continuous one parameter family of arcs in S^2 as in (iii) and if $\gamma(1)$ is independent of t then $\widehat{\gamma}(1)$ is also independent of t.

Properties (i)-(iii) go back to Caratheodory. An excellent modern exposition can be found in Mather's paper [M]. In particular see §17 of [M] for a discussion of accessible points and §18 for property (iv).

A properly embedded ray R in $\tilde{S}$ that is contained in V_n determines an arc γ as in (iii) with $\gamma(1) = \infty$. Considering the rays R and $G(R)$ and applying item (ii) of

Lemma 2.6 we obtain γ and $\widehat{G}(\gamma)$ to which Property (iv) applies. This implies that $\widehat{\gamma}$ and $\widehat{G}(\widehat{\gamma})$ converge to the same prime end $\widehat{\infty}$ which is evidently fixed by $\widehat{G}$.

Now choose a properly embedded ray R_1 in H_{n+1} with initial endpoint in K_n and note that R_1 is disjoint from the frontier of H_n. Since K_n is bounded, R_1 intersects the frontier of K_n in some first point z which is necessarily in the frontier of V_n. The initial segment γ_1 of R_1 that terminates at z satisfies the hypotheses of (iii) with $\gamma(1) = z$. Let $\hat{\gamma}$ be the associated path in D and let $\hat{z} \in \partial D$ be the prime end $\hat{\gamma}(1)$.

Our proof is by contradiction. Let m be the smallest natural number with $G^m(K_n) = K_n$ and assume $m \neq 1$ and $G(\bar{K}_n) \cap \bar{K}_n = \emptyset$. Then $\hat{z}, \widehat{G}(\hat{z}), \widehat{G}^m(\hat{z})$ and $\widehat{G}^{m+1}(\hat{z})$ are distinct and in this order on $\partial D \setminus \{\widehat{\infty}\}$ (oriented so that $\hat{z} < \widehat{G}(\hat{z})$). Since $G^m(K_n) = K_n$ the initial endpoints of the $\widehat{\gamma}_1$ and $\widehat{G}^m(\widehat{\gamma}_1)$ can be joined in K_n to form an arc $\widehat{\beta}$ with interior in K_n and endpoints $\hat{z}$ and $\widehat{G}^m(\hat{z})$. Therefore $\widehat{G}(\widehat{\beta})$ has one endpoint $\widehat{G}(\hat{z})$ and the other $\widehat{G}^{m+1}(\hat{z})$. It follows that $\widehat{G}(\widehat{\beta}) \cap \widehat{\beta} \neq \emptyset$ and indeed the points of intersection must lie in the interior of these arcs in contradiction to the assumption that $G(K_n) \cap K_n = \emptyset$. We conclude that $m = 1$ and hence that $G(K_n) = K_n$. □

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